Problems for 232A*

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June 11, 2020

1 Relations, Preferences, Utility

1.1 Extreme relations

Lots of political science and political economy uses the notion of an "ideological spectrum" represented by the real numbers $\mathbb{R}$, where higher numbers mean more conservative.[1] In some contexts it is useful to define an ideology of 0 as "neutral", and so having a negative ideology means liberal and positive means conservative.

We can define a relation on $\mathbb{R}$ called E, where $x\,E\,y$ if and only if $|x| \geq |y|$. That is, $x\,E\,y$ means that x is weakly more "extreme" than y.

i. Is the E relation a weak preference relation (i.e., complete and transitive)?

ii. Now consider the "strictly more extreme" relation SE, where $x\,SE\,y$ if and only if $|x| > |y|$. Is this a weak preference relation?

iii We might sometimes want to include more dimensions on our ideological spectrum. With two dimensions, we typically use $\mathbb{R}^2$, or the set of all order pairs of real numbers.

*Thanks to Andrew McCall, David Foster, Jieun Kim, and many students at Cornell and Berkeley for corrections and feedback. Email me (andrew.little@berkeley.edu) if you see any remaining mistakes

[1] It would be mathematically fine to define higher numbers as more liberal, but since we often us "right" to mean conservative it makes sense to place people who are more conservative farther to the right.

(Visually, think of the typical Cartesian plane). One proposal to make comparisons here would be to call one policy/politician more conservative than another if they are higher on at least one dimension. That is, define the relation C as $(x_1, x_2)\, C\, (y_1, y_2)$ if and only if $x_1 \geq y_1$ or $x_2 \geq y_2$. Is the C relation complete? Transitive?

iv. What if we define the "more conservative than" relation to mean "more conservative on all dimensions", i.e., $(x_1, x_2)\, C\, (y_1, y_2)$ if and only if $x_1 \geq y_1$ **and** $x_2 \geq y_2$. Is the C relation complete? Transitive?

v. You should have found that neither definition in parts iii or iv is both complete and transitive. Propose another way to define the "more conservative than" relation which does meet both requirements.

1.2 Transforming utility functions

A decision-maker makes a choice $s \in S$, with preferences represented by utility function $u(s)$.

i. Consider a second utility function that takes the utility given by u and doubles it. That is, $v(s) = 2u(s)$. Does v represent the preferences of the decision-maker as well? Why or why not?

ii. How about $v(s) = -u(s)$?

iii. How about $v(s) = 2u(s) - 5$?

iv. How about $v(s) = u(s)^2$?

v. What properties of v will guarantee that it represents the preferences in the same fashion as u? More precisely, can you give a condition such that v represents the decision-makers preferences if and only if that condition holds?

1.3 Learning about preferences

A dictator is deciding between some economic policies. The policies will affect his personal income, and the median income of the citizens. Suppose we know that his preferences can be represented with a utility function of the following form:

$$u(s, m) = s + am$$

where s is his own income ("self"), m is the median income of the citizens, and $a \geq 0$ measures how much he cares about the citizens (due to altruism, wanting to avoid a rebellion, etc.)

i. The dictator's advisors give him two options: Policy 1 ("major expropriation") will result in incomes $(s, m) = (1000, 1)$. Policy 2 ("minor expropriation") will result in $(s, m) = (800, 2)$. If the dictator prefers policy 2, what can we infer about the value of a? (These incomes could be in, say, thousands of dollars. Or dinars. Or bitcoin. Think through why it doesn't matter.)

ii. After the dictator tells us he likes policy 2, an advisor comes up with a better way to enact major expropriation, call this policy 3, which will result in incomes $(1200, 1)$. The dictator says he likes this even better than policy 2. Combining this with the information from part i, what do we now know about a?

iii. After implementing policy 3, the people are unhappy and overthrow the dictator for a new leader with more competent economic advisors. The new dictator has a utility function of the form:

$$u_2(s, m) = s + a_2 m$$

for some $a_2 \geq 0$. In office, the new dictator can choose between any of the 3 previous policies, but his brilliant economic team also offers and option (policy 4) which gives incomes $(1200, 3)$.

The new dictator prefers policy 4 to any of the previous ones. From this preference, what can we infer about a_2?

iv. Suppose we know a decision-maker has a utility function of the form:

$$u(x, y) = x + ay,$$

where $x \geq 0$, $y \geq 0$, and a is either equal to 1/2, 1, or 2. Find a set of three policy choices (x_1, y_1), (x_2, y_2), and (x_3, y_3) such that if we confront the policy maker with these three options, we can figure out what his value of a is from the choice.

2 Expected Utility

2.1 Expected utility and normalizing

A policy-maker can decide between accepting the status quo (outcome Q), or adopting a new policy which could fail (outcome F) or succeed (outcome S). If adopting the new policy, $Pr(S) = p$ and $Pr(F) = 1 - p$.

For now, assume that the utilities for each outcome are $u(F) = -1$, $u(Q) = 0$, and $u(S) = v > 0$.

i. Derive an inequality which shows that the new policy is rationalizable if and only if p is above a threshold.

ii. Now, let's keep the assumption that $u(Q) = 0$, but make no assumptions on the other utilities other than $u(F) < 0 < u(S)$. Derive another inequality to show for what values of p the new policy is rationalizable.

iii. Show that if we define $v = \frac{u(S)}{-u(F)}$, i.e., the relative gain from the policy succeeding (versus the status quo) to the loss from the policy failing, then we get the same inequality derived in part i. In words, why was it ok to assume $u(F) = -1$?

iv. Now let's relax the assumption that $u(Q) = 0$, while still maintaining that $u(F) < u(Q) < u(S)$. Derive the threshold in p where the new policy is rationalizable. Show that if $v = \frac{u(S)-u(Q)}{u(Q)-u(F)}$ this is equivalent to the threshold we found in part i.

2.2 Imperfect Experts

One way to learn about whether policies will work is to consult experts. But experts are sometimes wrong. As in the previous problem, suppose a new policy succeeds with (prior) probability p, and gives utility $u(S) = v > 0$ if succeeding and $u(F) = -1$ if failing. If the policymaker keeps the status quo, her utility is $u(Q) = 0$.

An expert, who is not a strategic actor, sends a message s or f before the policy maker decides.

i. Suppose the expert's message is correct (i.e., s if the policy will succeed, and f if it will fail) with probability $q > 1/2$. So with probability $1 - q$, the message is s when the policy will fail and f when the policy will succeed. Use Bayes' rule to find the probability that the project will succeed conditional on the expert saying s. What happens as $q \to 1/2$ and $q \to 1$?

ii. Similarly, find the probability that the project will succeed if the expert says f? What happens as $q \to 1/2$ and $q \to 1$.

iii. Suppose $p = 1/2$. How large does q have to get before the policymaker always follows the advice of the expert? (Your answer should depend on v.) Show that a sufficiently good expert (q close to 1) is always followed, and as long as $v \neq 1$, a sufficiently poor expert (q close to 1/2) is always ignored for at least one message.

3 Normal Form Games, Solution Concepts

3.1 Modeling the 25th Amendment

The 25th Amendment of the U.S. Constitution states "Whenever the Vice President and a majority of either the principal officers of the executive departments or of such other body as Congress may by law provide, transmit to the President pro tempore of the Senate and the Speaker of the House of Representatives their written declaration that the President is unable to discharge the powers and duties of his office, the Vice President shall immediately assume the powers and duties of the office as Acting President."

Propose a Normal Form Game (actors, strategies, utilities) which could plausibly represent the relevant actors deciding whether to invoke the 25th amendment. (Don't worry about getting the constitutional details right.) For utilities, what might represent someone who is truly loyal to the president? Someone who is an opportunist and wants to support the winning side?

3.2 The Pareto Dominance Relation

Recall a relation can be any way to compare two elements of a set. So, if S is the set of outcomes in a game, we can define the D relation to mean that xDy if and only if x Pareto dominates y.

i. Show that the D relation is transitive (i.e., if xDy and yDz, then xDz)

ii. Show (by finding a counterexample) that D is not necessarily comparable (i.e., it can be the case that $x \not{D} y$ and $y \not{D} x$).

iii Take any normal form game with players $N = \{1, ..., n\}$, strategies S_i, and utilities $u_i(s)$. Suppose a social planner (external to the game) has a preference relation P_{sp} such that $s^1 P_{sp} s^2$ if and only if

$$V(u_1(s^1), ..., u_n(s^1)) \geq V(u_1(s^2), ..., u_n(s^2))$$

for some function V (which maps all the individual utilities to a single real number). Show that (1) for any V, this is a proper preference relation (i.e., it is comparable and transitive), and (2) for any game and outcome $s^* \in S$, there exists a V where $s^* P_{sp} s'$ for any s'. What does this mean?

3.3 Condorcet Strategy Profiles

Three members of a committee ($N = \{1, 2, 3\}$) have to vote on a potential new policy. There are three options $S_i = \{A, B, C\}$. If the players unanimously pick the same policy, it is implemented, and if not the committee is disbanded with no new policy set. Let $v_i(J) > 0$ be the utility for player i if policy J is chosen unanimously, and all get a utility of 0 if the vote is not unanimous.

i. Show that for any values we choose for the $v_i(J) > 0$ payoffs, (1) no outcome is a strictly dominant strategies equilibrium, and (2) every outcome is a no strictly dominated strategies equilibrium

ii. Show that any non-unanimous vote is not Pareto efficient. Find some values of the $v_i(J)$ variables where there is (1) a unique Pareto efficient outcome, (2) two Pareto efficient outcomes, and (3) three Pareto efficient outcomes.

iii. We may find it disappointing that Pareto efficiency might identify multiple policies. (This could be worse in a voting game with lots of players, where any individual liking a policy best ensures a unanimous vote for it it is Pareto efficient). So let's try a stronger solution concept which requires a majority of the voters like a policy being implemented more than other outcomes. Say a strategy profile s^* is a *Condorcet winning profile* if and only if for all $s' \in S$, $u_i(s^*) \geq u_i(s')$ for at least half of the i's (in this case, 2). Show that if $v_1(A) > v_1(B) > v_1(C)$, $v_2(B) > v_2(C) > v_2(A)$, and $v_3(C) > v_3(A) > v_3(B)$, there is no Condorcet winning profile.

iv. Find a set of values for the $v_i(J)$s where (1) a unanimous vote for each policy is Pareto efficient, but (2) there *is* a unique Condorcet strategy profile.

3.4 Costly Sucking Up

Two students are deciding how much to suck up to their professor. Their strategy sets are $S_i = \{1, 2, ..., 10\}$ ("on a scale from 1 to 10, how great is this class?"). They would like to suck up a bit more than their colleague but not too much more, and also feel a small sense of shame from sucking up. To formalize, their utility is:

$$u_i(s_i, s_{-i}) = -|s_i - (s_{-i} + 1)| - \epsilon s_i$$

where $0 < \epsilon < 1$ represents the cost of sucking up.

i. What is/are the sum utility maximizing outcome(s)? (Hint: start by showing that $|s_1 - (s_2 + 1)| + |s_2 - (s_1 + 1)|$ is at least 2.)

ii. Show that any outcome where $s_1 > 1$ and $s_2 > 1$ is Pareto dominated by $(s'_1, s'_2) = (s_1 - 1, s_2 - 1)$.

iii. Show that any outcome where one player chooses $s_i = 1$ and the other chooses $s_i > 2$ is Pareto dominated by $s_i = 1$ and $s_{-i} = 2$.

iv. Given ii and iii, what are the three possible Pareto efficient outcomes? Show that these remaining three outcomes are in fact Pareto efficient.

v. What is the best response correspondence for each player?

vi. What outcomes are a best response to something equilibrium (as defined on the handout)? Compare this to your answer to ii-v

3.5 Altruistic PD

One way to model altruism is to assume that the player's utilities are equal to their "objective" payoff plus some multiplier times the utility of the other player(s).

Let's study a two player prisoners' dilemma, with strategy sets $S_i = \{C, D\}$ and *objective* payoffs:

$$x_i(s_i, s_{-i}) = \begin{cases} 3 & s_i = D, s_{-i} = C \\ 2 & s_i = C, s_{-i} = C \\ 1 & s_i = D, s_{-i} = D \\ 0 & s_i = C, s_{-i} = D \end{cases}$$

and utility functions (which actually determine how we solve the game):

$$u_i(s_i, s_{-i}) = x_i(s_i, s_{-i}) + \alpha x_{-i}(s_i, s_{-i})$$

where $0 \leq \alpha \leq 1$ measures how much players care about the others payoffs.

i. Represent this game in table form.

ii. What is the strict dominant strategy equilibrium (if any) to this game for all allowed (i.e., between 0 and 1) values of α?

iii. For what values of α is (C, C) the unique Pareto efficient outcome?

iv. What would it mean if $\alpha > 1$? Show that if α is sufficiently high, then (C, D) and (D, C) are Pareto efficient. How high does α have to be for this to be true? In words, why does being "too altruistic" make the exploitative outcome efficient again?

4 Nash Equilibrium

4.1 Comparing Solution Concepts

Consider the following combinations of solution concepts for normal form games. Exactly one of them is not possible. Identify which is not possible and why, and for all of the others give an example (either in table form, or directly give the players/strategies/utilities) where it holds.

A. There is a unique Pure Strategy Nash Equilibrium which is Pareto efficient.

B. There is a Pure Strategy Nash Equilibrium which is not Pareto efficient.

C. Every outcome is Pareto Efficient, but none are a Pure Strategy Nash Equilibrium.

D. There is a unique strictly dominant strategies equilibrium, and more than one no strictly dominated strategies equilibrium.

4.2 Costless Voting

Here is a very simple "costless voting" game. There are $n > 2$ players who all choose $v_i \in \{0, 1\}$, with utility:

$$u_i = \begin{cases} 1 & \sum v_i \geq n/2 \\ 0 & o/w \end{cases}$$

i. Show that there is a Nash Equilibrium to this game where $v_i^* = 1$ for all i, and another Nash Equilibrium where $v_i^* = 0$ for all i.

ii. What is the complete set of Nash Equilibria to this game? (To reduce cases, you can assume that n is even.)

iii. The NE where all choose $v_i^* = 0$ may seem odd, because there is never anything to gain by choose $v_i = 0$ over $v_i = 1$, and choosing $v_i = 1$ might strictly increase i's utility. A way to formalize this notion (slightly different than what is on the definitions handout; unfortunately there is no consistently-used definition here) is that a strategy s_i is weakly dominated if there exists a $s' \in S_i$ such that $u_i(s_i, s_{-i}) \leq u_i(s'_i, s_{-i})$ for all $s_{-i} \in S_{-i}$, and $u_i(s_i, s_{-i}) < u_i(s'_i, s_{-i})$ for some $s_{-i} \in S_{-i}$. Show that for each player, choosing $v_i = 0$ is weakly dominated.

iv. A common "refinement" to Nash Equilibrium in voting games is to assume that players don't use weakly dominated strategies. Does this game have NE where players don't use weakly dominated strategies? If so, is it unique?

4.3 Simultaneous Bargaining and War

Two players ($N = \{A, B\}$) are eyeing a resource with value $v > 0$. Both simultaneously make a claim on a portion of the prize $x_J \in [0, v]$. If the sum of the claims is less than or equal to the total prize, each gets a payoff equal to their claim. If their claims are greater than the prize, they fight over the prize, with "war" payoffs $w_A \geq 0$ and $w_B \geq 0$. So we can write the utility functions:

$$u_J(x_J; x_{-J}) = \begin{cases} x_J & x_J + x_{-J} \leq v \\ w_J & o/w \end{cases}$$

i. Derive the best response functions for the players. (Hint: there is sometimes a unique best response, and sometimes a range of choices which are a best response.)

ii. What are the Nash Equilibria to this game if $w_A + w_B < v$?

iii. What are the Nash Equilibria to this game if $w_A + w_B > v$?

iv. Which of the NE identified in part ii are Pareto Efficient?

4.4 Discrete Political Competition

Two major parties L and R are competing in an election, and can choose from platforms $x_J \in \{l, m, r\}$. If both pick the same platform, that platform wins. If they pick different platforms, then m wins no matter what. (To motivate this, think that if one picks m, then m wins. If one picks l and the other picks r, then a small (nonstrategic) centrist party wins and implements m). Formally, the winning platform is:

$$w = \begin{cases} l & s_L = s_R = l \\ r & s_L = s_R = r \\ m & \text{otherwise} \end{cases} \tag{1}$$

The utilities as a function of w are:

$$u_L = \begin{cases} -v & w = r \\ 0 & w = m \\ v & w = l \end{cases}, \text{ and } \quad u_R = \begin{cases} -v & w = l \\ 0 & w = m \\ v & w = r \end{cases} \tag{2}$$

i. In words, what do these utilities mean about the parties preferences?

ii. Represent this game in table form (i.e., a 3x3 table)

iii. What are the NE to this game?

4.5 Political Competition with Entry

Here is a variant of the political competition model, with two changes. First, there are now three parties: $N = \{1, 2, 3\}$. Second, in addition to choosing a platform, they also decide whether to run in the first place. Formally, we can represent their strategy set as $S_i = \{\text{no run}\} \cup \mathbb{R}^+$ ("don't run", or the platform if you run).

Assume there is a large and odd number of voters (at least 7 to make things simpler) with distinct ideal points, and hence there is a unique median ideal point m. As long as at least one party runs, voters vote for the party with the closest platform to their ideal point, and split their votes equally between candidates if there is a tie. That is, if there are two parties which a voter is indifferent between, each get a half vote, and if there are three parties a voter is indifferent among each gets a third of a vote.

Let v_i be the votes for candidate i. A candidate "wins" (where this includes the case of ties; it will be clear when the utility is presented) if $v_i \geq v_{-i}$ for all v_{-i}. and loses otherwise. Let w be the number of candidates that enter and win. The utilities are:

$$u_i = \begin{cases} 1/w - k & \text{enter and win} \\ -k & \text{enter and lose} \\ 0 & \text{otherwise} \end{cases}$$

For some $k \in (0, 1/3)$.

Our goal is going to be to show that this game has *no* pure strategy Nash equilibrium. To demonstrate this, we will show that any "class" of equilibrium leads to a contradiction.

i. Show that in any NE, at least two parties enter the race (i.e., at most one chooses "no run")

ii. Show that in any NE where multiple candidates enter, they must tie

iii. Show that there is no NE where two candidates enter (and tie)

iv. Show that there is no NE where all three candidates enter and choose the same position.

v. Show that there is no NE where all three candidates enter and do not choose the same position.

vi. Parts i-v imply there is no pure strategy Nash Equilibrium (you can assume v is true even if you didn't prove it). Why?

5 Nash Equilibrium with Calculus

5.1 A Brief Calculus Diversion

A frequently employed utility function in a lot of applied models looks like this:

$$u(x) = -w_1(x - t_1)^2 - w_2(x - t_2)^2$$

where x is the choice variable, and t_1 and t_2 represent two "targets" which the decision-maker wants the choice to be close to. For example, if a legislator wants to advocate policies which he likes but also policies popular with her constituents, t_1 might represent her ideal policy and t_2 the average or median constituent's ideal policy.

What is the value of x which maximizes this function? How does increasing w_1 or w_2 affect the optimal choice? (You can use the quadratic maximizing trick from class or solve using other techniques).

5.2 Infinite Love for Dear Leader

This is a game played between two citizens who are asked how much they love their dear leader. Both players have a common level of love for the dear leader $x \in \mathbb{R}$ (i.e., can be any negative or positive number). The citizens dislike lying, but also want to say they love the dear leader more than the other citizen. The specification of the game is:

Players: $N = \{1, 2\}$

Strategy: $s_i \in \mathbb{R}$ (report of love for leader)

Preferences: $u_i = -b(s_i - x)^2 - (s_i - (s_{-i} + a))^2$, where $b > 0$ and $a > 0$

Note that the first term captures the fact that the citizens want to report the truth: their payoff from this term is decreasing in the distance between s_i and x. This term is scaled by b, so higher b means the citizens care more about reporting their true feelings. The second term implies that the citizen also wants to make a guess close to $s_{-i} + a$, which captures the fact that they want to "out love" the other citizen (though not TOO much, as guessing higher than $s_{-i} + a$ leads to lower payoffs from this term as well). If a is large, the citizen wants to say the love the leader much more than the other citizen.

i. Find the Nash Equilibrium to this game. Hint: use best response functions and your answer to the previous question.

ii. What are the comparative statics on b, x, and a. That is, as these parameters go up, how does the reported love for the dear leader change? How does the strategy differ from the citizen's true feelings (i.e., x)?

iii. What happens to the equilibrium strategies when b is very small? That is, what if the citizens care very little about reporting their true feelings?

iv. What is the citizens' equilibrium payoff? (You can calculate this by plugging in the equilibrium strategies into the utility function: e.g., $u_1(s_1^*, s_2^*)$). How does changing x, a, and b change the equilibrium payoff (i.e., what are the comparative statics)? Show that the equilibrium outcome is not Pareto efficient.

5.3 Bring on the costly regulation

(Inspired by "The Enemy of My Enemy: When Firms Support Climate Change Regulation " by Amanda Kennard.)

In class we studied a variant of the Cournot bribing game where the bureaucrats have different marginal costs. To refresh, there are two bureaucrats $N = \{1, 2\}$, who both pick a quantity of bribes to supply $q_i \geq 0$. The resulting price of a bribe is $p = \overline{p} - q_1 - q_2$, and the utilities are:

$$u_i(q_i, q_j) = pq_i - c_iq_i \tag{3}$$

where $c_i < \overline{p}$.

i. As we did in class, what is the (potential) NE to this game where both bribe supplies are positive ($q_i^* > 0$)?

ii. As the parenthetical indicates, our analysis only looked for equilibria where the best responses are positive. Show that for some values of $c_1 < \overline{p}$ and $c_2 < \overline{p}$ one of the players is actually choosing a negative quantity in the (potential) NE identified in part i. Intuitively, why is this the case? What additional restrictions can we put on c_1, c_2, and $\overline{p}$ to prevent this from happening? [OPTIONAL: Find the NE when these restrictions are not met.]

iii. Now let's ask what happens when the government proposes a regulation which makes bribes costlier for both players. In particular, assume we can write the costs as:

$$c_1 = rc \tag{4}$$

$$c_2 = rcd \tag{5}$$

where $c > 0$ is the "baseline cost", r is the level of regulation, and $d > 1$ is the degree to which bureaucrat 2 is "disadvantaged" in supplying bribes (in the sense that it is more costly for her). Show that if d is sufficiently high, it is possible for the quantity of bribes produced by bureaucrat 1 to be increasing in how strictly bribes are regulated (i.e., q_1^* is increasing in r). In words, why is this possible?

vi. Show that if d is "too high", it is also possible to violate the constraint identified in part ii. Find some parameter values of r, d, c, and $\overline{p}$ which don't violate the constraints from part ii, and where a small increase in r would increase the bribes supplied by 1 (i.e., meets the requirement you identified in part iii.)

6 Mixed Strategies

6.1 Terrorism and Counterterrorism

Consider the following game between a terrorist organization and the state. The terrorist chooses between attacking one of two sites, and the state can defend one site or the other. The terrorist organization wants to attack the site that the state does not defend, and the state wants to defend the site that is attacked. This can be represented with the following payoff matrix

		S: 1	S: 2
T	1	$0, 0$	$v_1, -k_1$
	2	$v_2, -k_2$	$0, 0$

That is, both sides get a payoff of 0 if the state defends the correct site. If the terrorist successfully attacks site 1, they get a payoff of $v_1 > 0$ and the state gets $-k_1 < 0$. If the terrorist successfully attacks site 2, the analogous payoffs are v_2 and k_2.

i. Show there is no NE where either player uses a pure strategy.

ii. Given i, what does the Nash Existence Theorem tell us?

iii. Find the mixed strategy NE to this game.

iv. In the mixed strategy equilibrium, how do the v and k terms affect the players' behavior? (What principle that we discussed in class does this illustrate?)

6.2 General Best Responses

Take a general 2x2 game with players $N = \{1, 2\}$ and actions $S_i = \{A, B\}$, and utilities $u_i(s_1, s_2)$. Let $p_i = Pr(s_i = A)$.

i. What is the best response correspondence for player 1? Hint: consider cases based on whether $EU_1(A, p_2) < EU_1(B, p_2)$, $EU_1(A, p_2) = EU_1(B, p_2)$, or $EU_1(A, p_2) > EU_1(B, p_2)$

In our proof that an MSNE always exists in a 2x2 game, we relied on a claim that at least one of the following three is true (with an analogous claim for player 2):

1. $EU_1(A, p_2) \geq EU_1(B, p_2)$ for all p_2

2. $EU_1(A, p_2) \leq EU_1(B, p_2)$ for all p_2

3. There exists a $p_2^* \in (0, 1)$ such that $EU_1(A, p_2^*) = EU_1(B, p_2^*)$

To prove this, we will show that if 1 and 2 are both false, then 3 must be true.

ii. Show that $EU_1(A, p_2) < EU_1(B, p_2)$ for some p_2 (i.e., 1 is false) if and only if $u_1(A, A) < u_1(B, A)$ OR $u_1(A, B) < u_1(B, B)$.

iii. Show that $EU_1(A, p_2) > EU_1(B, p_2)$ for some p_2 (i.e., 2 is false) if and only if $u_1(A, A) > u_1(B, A)$ OR $u_1(A, B) > u_1(B, B)$.

iv. Combining ii-iii, show that if 1 and 2 are false it must be the case that (I) $u_1(A, A) < u_1(B, A)$ and $u_1(A, B) > u_1(B, B)$ or (II) $u_1(A, A) > u_1(B, A)$ and $u_1(A, B) < u_1(B, B)$. In words, what do these two conditions mean?

v. Finally, solve for the value of p_2 which solves $EU_1(A, p_2) = EU_1(B, p_2)$, and show that it lies within $(0, 1)$ if either condition (I) or condition (II) from part iv holds. [Hint 1: let $d_A = u_1(A, A) - u_1(B, A)$ and $d_B = u_1(A, B) - u_1(B, B)$. Hint 2: for any x and y, $x/y \in (0, 1)$ if and only if $0 < x < y$ or $y < x < 0$.]

6.3 Mixed Entry into Political Competition

In problem 4.5 we analyzed a game where three parties chose to enter a political competition or not, and if so what platform to offer. We found that there was no pure strategy Nash Equilibrium, so maybe there is one in mixed strategies.

A natural place to start is an equilibrium where each party enters with probability q, and if entering picks the platform of the median voter.

i. Let's first see what happens if we drop the platform choice and assume that parties only decide whether to enter or not. If n parties enter, each has a $1/n$ chance of winning. As before, the cost of entering is $k > 0$. If all parties enter with probability q, write a condition for each to be indifferent between entering and not.

ii. Solve this equation for q, which tells us the mixing probability required to make all indifferent as a function of k. Show that there is a value of $q \in (0, 1)$ which solves this equation if and only if $1/3 \leq k \leq 1$. [Hint/warning: requires finding the roots of a quadratic equation]

iii. If we allow the parties to select the platforms, then the analysis above is one condition for there to be a MSNE where all enter with probability q but always pick the ideal platform of the median voter if entering. However, we also need to check that it would not be profitable to enter and pick a different platform. One can show that if there are enough voters, the best possible alternative platform is something between the median and the next close voter to the left or right. Further, if doing so, the party not at the median wins if no other party enters or if two other parties enter (as the other two split the votes of the median and those to the opposite side of the deviator), but if one other party enters the deviator loses (by the standard logic). Show that if q is sufficiently low, this constraint is met.

iv. Combine parts ii and iii to give a range of k where each party is indifferent between staying out and entering and offering the platform of the median voter, and prefers either of these options to entering and proposing a different platform.

6.4 Mixed Strategies in Coordination

Two players pick from a choice set $S_i = S = \{1, 2, ..., m\}$. If they pick the same number, they get a payoff of 1. If they pick different numbers, they get 0.

i. Let p_i^j be the probability that player i picks choice j. Show that there is a MSNE where all options are picked with positive probability.

ii. Show that for any subset $A \subseteq S$, there is a MSNE where the players play all of the choices in A but none of the choices outside of A.

iii. Now suppose if both player choose option j they get payoff $v^j > 0$. What are the MSNE to this game? (Hint: define $K = \sum_{i=1}^{m} 1/v^j$.)

7 Evolutionary Game Theory

7.1 Generalized Evolutionary Battle of the Berkeley Bowls (GEBoBB)

Here is a general version of the evolutionary Battle of the Berkeley Bowls. First let's analyze the Normal Form version of the game. Both players choose either C ("close") or F ("far"). If both choose close, they go to different stores and get utility zero. If both choose far, they pay a cost to travel $k > 0$ and go to different stores, giving utility $-k$. If one chooses close and the other chooses far, the one who picked close gets $v > 0$ and the one who picked far gets $v - k$, where $k < v$.

i. What are the (pure and mixed strategy) NE to this game?

ii. Now consider the evolutionary version of this game, where the fitness functions are the same as the utilities in the normal form version. What is the expected fitness to each type as a function of the proportion of C players in the population (call this p)?

iii Plot the fitness of each type as a function of p. For what values of p will the proportion of C types increase? Decrease?

iv. What is the evolutionarily stable population? How does this compare to the NE of the normal form game?

v. How does changing v and k affect the NE and evolutionarily stable population?

8 Extensive Games

8.1 Some Practice with Extensive Games

Consider the following extensive game:

At the initial node, player Y makes a decision with action set $A = \{1, 2, 3, 4\}$.

If Y chooses 1 or 2, this is followed by a node where player Z chooses from action set $A = \{5, 6\}$, both of which lead to a terminal node.

If Y chooses 3, this is followed by a node where Z chooses from action set $A = \{6, 7\}$, both of which lead to a terminal node.

If Y chooses 4, this leads to a node where Y moves again with action set $A = \{1, 2, 3, 4\}$, which all lead to terminal nodes.

i. Draw the game tree version of this extensive game (don't worry about writing out utilities, which were not specified)

ii. What are the possible information partitions for player Z? In words, what is the difference between the strategic situation modeled by these two information partitions?

iii. What are the possible information partitions for player Y? In words, what is the difference between the strategic situation modeled by these two information partitions? (Hint: one of them is kind of weird.)

iv. Which of the possible information partitions (for both players) leads to the most subgames? The fewest subgames?

v. Draw the game tree version of the game with the "coarsest" possible information partition (i.e., lumping together as many nodes as possible). (Hint: use the "circle nodes in the same information set" convention.) How many strategy profiles are there in this version of the game?

8.2 The Centipede Game and (SP)NE

Here is a version of the famous "centipede game". There is a pot of money between two players, and they take turns either grabbing (G) the pot or waiting (W). Every time a player waits, the amount in the pot goes up by 1.

Suppose the pot starts with $x > 0$, and there are three rounds. If the players wait for all 3 rounds, they both get 1/2 of the pot. If a player grabs, they get 2/3 of what is in the pot, with the other player getting the remaining 1/3. So, the payoffs at each terminal node are:

- G: $(\frac{2}{3}x, \frac{1}{3}x)$
- W, G: $(\frac{1}{3}(x+1), \frac{2}{3}(x+1))$
- W, W, G: $(\frac{2}{3}(x+2), \frac{1}{3}(x+2))$
- W, W, W; $(\frac{1}{2}(x+3), \frac{1}{2}(x+3))$

The game has perfect information.

i. Write this as a normal form game table (hint: player 1 should have 4 strategies and player 2 should have 2 strategies).

ii. What are the NE to this game when $x < 1$? When $1 < x < 5$? (Don't worry about the case where $x = 1$ or $x \geq 5$)

iii. Draw the extensive form of the game (with perfect information). What are the subgames?

iv. What is the SPNE if $x < 1$? If $1 < x < 5$.

v. For what values of x is there an NE *outcome* which is not a SPNE outcome?

8.3 Loyalty and Competence

An executive (E) needs to decide who to appoint to head an agency. She has two options, who we will call Loyal (L) and Competent (C). As will be formalized, the competent choice will do a better job of implementing policy. Let $H \in \{L, C\}$ represent this choice.

Whichever head is chosen then picks a policy $P \in \{A, U\}$, where A means "aligned" and U "unaligned".

The president wants the head to pick policy A, but also prefers to appoint the competent head. To formalize, let E's utility be:

$$u_E = w\mathbf{1}_{P=A} + \mathbf{1}_{H=C}$$

where $w > 0$ represents how much he cares about policy vs competence.

Both potential heads prefer to get hired. The loyal one prefers policy A, while the competent one prefers U. Formally:

$$u_L = \mathbf{1}_{H=L} + w\mathbf{1}_{P=A}$$

$$u_C = \mathbf{1}_{H=C} + w\mathbf{1}_{P=U}$$

i. What is the SPNE and SPNE outcome to this game if $0 < w < 1$?

ii. What is the SPNE and SPNE outcome if $w > 1$?

iii. What other outcomes can be reached in a NE when $w > 1$?

iv. Suppose E also has an option to hire a third head, who is both loyal and competent, i.e., has a utility $U_{LC} = w\mathbf{1}_{P=A} + \mathbf{1}_{H=LC}$ and $u_E = w\mathbf{1}_{P=A} + \mathbf{1}_{H=C \text{ or } H=LC}$. Show that in any SPNE, E hires LC, but there is still an NE where a different head is appointed.

v. Suppose that the potential loyal and competent hire is the executive herself, with the same utility assigned to her in part iv. Is hiring someone else still a possible outcome in an NE?

9 Bargaining

9.1 Presidential Vetoes with Discrete Bills

A legislature l is debating a set of bills $B = \{1, 2, .., m\}$, $m > 1$. The legislature can either send a $b \in B$ to the president to sign, or send no bill, write this $b = 0$. The President observes b, and then decides whether to veto ($v = 1$) or sign ($v = 0$) the bill.

Let $u_l(b)$ and $u_p(b)$ be the utilities to the legislature and president if b is sent and not vetoed. Assume $u_l(0) = u_p(0) = 0$ (i.e., normalize the "status quo" payoffs to 0). To simplify, assume that for any $b > 0$, $u_l(b) \neq 0$ and $u_p(b) \neq 0$, and for any $b_i \neq b_j$, $u_l(b_1) \neq u_l(b_2)$ and $u_p(b_1) \neq u_p(b_2)$ (i.e., neither actor is indifferent between any two bills.)

i. Suppose if the president vetoes, the utilities are also both 0. Explain what this assumption means in words.

ii. What is the set of possible strategies for the legislature? For the president? (As a hint, it might be useful to write $v(b)$ to represent what the president chooses to do when given bill b.)

iii. For large m this will result in a large strategy set for the president and hence fully describing every Nash Equilibrium will be a pain. However, it is simpler to ask whether certain *outcomes* can be a NE.

Show that given the assumptions so far, there is always a NE where the outcome is the legislature not sending a bill $b^* = 0$. That is, find a strategy for the president $v^*(b)$ and a strategy for the legislature b^* where these strategies are mutual best responses.

iv. Show that there is a NE where bill b^* is sent and not vetoed if $u_l(b^*) > 0$ and $u_p(b^*) > 0$. Again, be sure to specify the full strategy profile where this is the outcome and show that the players are choosing a best response.

v. What is the president's strategy in any SPNE?

vi. Find a condition for when the SPNE results in no bill being sent, or the bill sent being vetoed. Identify the bill which is sent and signed if this condition does not hold.

9.2 Bargaining with Altruism and Spite

Two people play an ultimatum game to determine how to divide a dollar. There is a proposer who makes an offer x, $0 \leq x \leq 1$. The responder observes this offer and then decides whether to accept ($a = 1$) or reject ($a = 0$). The twist is that each player cares not just about their monetary payoff, but that of their partner as well. First suppose their payoffs are:

$$u_P = a((1 - x) + bx)$$
$$u_R = a(x + b(1 - x))$$

where $b \geq 0$. To interpret, if the responder rejects, both get a utility of zero ($u_P = u_R = 0$). If the offer is accepted, the receiver gets x share of the dollar and the proposer keeps $1 - x$. However, the *utility* for both players is equal to their monetary payoff plus b times the payoff of the other player. (The standard ultimatum games is the case where $b = 0$).

i. Show that whenever $b > 0$, the responder accepts all offers in any SPNE.

ii. Find the SPNE if $b > 1$ (it is unique).

ii. Find the SPNE if $b = 1$ (there is more than one).

iii. Find the SPNE if $0 < b < 1$ (it is unique).

Now consider the case where our bargainers want the other player to get a *lower* payoff. To model this, keep the order of moves the same, but let the payoffs be:

$$u_P = a((1 - x) - sx)$$
$$u_R = a(x - s(1 - x))$$

for $s \geq 0$. So, higher values of s ("higher spite") mean the bargaining gets more disutility from his partner getting more money.

iv. What is the lowest offer which the responder accepts in an SPNE as a function of s? What happens and why if $s = 0$? $s = 1$? $s \to \infty$?

v. What is the largest value of x where the *proposer* prefers that the offer is accepted to being rejected?

vi. Find the (unique) SPNE if $s < 1$. How does the offer change as s increases?

vii. Show that when $s > 1$, there is no SPNE where an offer is accepted.

viii. At least when modeled this way, which is more likely to lead to even divisions in an ultimatum game: altruism or spite? (You can assume the case where b is exactly equal to 1 is unlikely)

9.3 Nixon Goes to the Ultimatum Game. (With Trump.)

In this problem set we will analyze a variant of the ultimatum game where one player (the "voter") gets to elect a representative to bargain on their behalf.

To start, consider the following game:

- Players: Negotiators 1 and 2, and a citizen (C_2) represented by negotiator 2
- Sequence of moves:
    1. C_2 picks either Nixon (N) or Humphrey (H) ($V \in \{N, H\}$) to be their negotiator
    2. Negotiator 1 makes offer $x \in [0, 1]$
    3. Negotiator 2 accepts ($a = 1$) or rejects ($a = 0$)

- The payoffs for C_2, 1, and 2 are:

$$u_{C_2} = ax$$
$$u_1 = a(1-x) + (1-a)b_1$$
$$u_2 = ax + (1-a)b_2$$

where $b_1 \in (0,1)$ and $b_2 = b_V$ where V represents the selected negotiator, with $0 < b_H < b_N < 1$. That is, the b_1 is the payoff to negotiator 1 if a deal is not made, and b_N is the payoff to negotiator 2 if it is Nixon while b_H is the payoff to negotiator 2 if it is Humphrey. (These payoffs can represent enjoying rejecting bad offers. Or, if rejecting leads to conflict, this could represent a "taste for fighting". So, $b_H < b_N$ means that Nixon gets a higher payoff from a deal not being made.[2]) The voter always gets a payoff of 0 if no deal is made.

i. Find the SPNE (and the corresponding outcome) to this game when $b_1 + b_N < 1$. That is, specify which offers are accepted by both types of negotiator 2, the offer that negotiator 1 makes to each negotiator type, and which negotiator the citizen selects.

ii. Find the SPNE (and the corresponding outcome) when $b_1 + b_N > 1$ but $b_1 + b_H < 1$.

iii. Suppose before C_2 picks their negotiator (where the choices are again N and H), a citizen in country 1 C_1 also gets to vote for their negotiator, say, either Trump (T) or Rubio (R). C_1's utility is $u_{C_1} = a(1-x)$. The payoffs for the other actors are the same as above, except now $b_1 = b_T$ if the voter picks trump and $b_1 = b_R$ if the voter picks Rubio. Find the SPNE if $b_T + b_N > 1$, but $b_R + b_N < 1$ and $b_T + b_H < 1$ (which also implies $b_R + b_H < 1$).

iv. Interpret these results. When is it valuable to send a "crazy" negotiator to bargain

[2] As the man himself explained it to his chief of staff "I call it the Madman Theory, Bob. I want the North Vietnamese to believe I've reached the point where I might do anything to stop the war. We'll just slip the word to them that, 'for God's sake, you know Nixon is obsessed about communism. We can't restrain him when he's angry—and he has his hand on the nuclear button' and Ho Chi Minh himself will be in Paris in two days begging for peace"

for you?

9.4 2 Dimensional Bargaining

Two countries choose pollution levels $x_1 \in [0,1]$ and $x_2 \in [0,1]$. Country i would like to pollute at level $x_i = 1$, and would like the other country pollute at $x_{-i} = 0$. Let their utilities be:

$$u_i(x_i, x_{-i}) = -(1 - x_i)^2 - x_{-i}^2$$

i. First suppose both countries decide their pollution levels simultaneously. What is the NE to this game?

ii. Draw a diagram which illustrates the set of outcomes which Pareto dominate the NE outcome. Hint: put x_1 on the x axis and x_2 on the y axis, and recall the circle centered at (a, b) with radius r is the set of points such that $(x_1 - a)^2 + (x_2 - b)^2 = r^2$.

iii. What is the set of outcomes which are Pareto Efficient?

iv. Now suppose the countries bargain over a pollution treaty in the following manner:

1. Country 1 makes an offer (x_1, x_2)
2. Country 2 observes the offer and decides to accept or reject.
3. If accepting, the proposal is implemented
4. If rejecting, both players simultaneously choose their own pollution levels as in part i

What is the set of offers which country 2 will accept in any SPNE (you can assume they accept offers when indifferent between accepting and rejecting)?

v. Locate on your diagram the offer which country 1 will make in a SPNE.

vi. Solve for the offer which country 1 makes algebraically

vii. How would your answers to iv-v change if country 2 makes the offer and country 1 accepts or rejects?

9.5 Nash Bargaining

Two players are bargaining over a policy x. If they reach an agreement, their utilities are $u_1(x)$ and $u_2(x)$. If they do not reach an agreement, their utilities are d_1 and d_2. If there exists an x' such that $(u_1(x') - d_1) > 0$ and $(u_2(x') - d_2) > 0$, then the *Nash Bargaining Solution* to this game is the policy x which maximizes:

$$(u_1(x) - d_1)(u_2(x) - d_2)$$

I.e., the product of the "surplus" for each player over their disagreement point. (Nash provided an axiomatic foundation for why this is a "reasonable" solution to expect. For those curious to read more, the wikipedia page on the NBS is pretty good: `https://en.wikipedia.org/wiki/Bargaining_problem`) If there is no x' such that $(u_1(x') - d_1) \geq 0$ and $(u_2(x') - d_2) \geq 0$, then there is no NBS: intuitively, if no deal makes both players better of than disagreeing, then we should predict disagreement

i. Take a bargaining model of war where $u_1(x) = x$, $u_2(x) = 1 - x$, $d_1 = p - c$, and $d_2 = 1 - p - c$, where $p \in [0, 1]$ and $c > 0$. What is the Nash Bargaining solution to this model? Which parameters affect the solution?

ii. Now take the payoffs from Little and Zeitzoff (2017),[3] where we call player 1 the proposer and 2 the responder: $u_1(x) = 2v - x$, $u_2(x) = x$, $d_1 = v - k + \beta_1$, $d_2 = v - k + \beta_2$. Show that the NBS exists if and only if $\beta_1 + \beta_2 \leq 2k$ (i.e., the same condition for a SPNE where a deal is made)

iii. Show that when it exists, the NBS gives the same payoff as the average payoff when

[3] `http://andrewtlittle.com/papers/little_zeitzoff_btcep_web.pdf`

two players are assigned in the proposer role and responder role with probability 1/2.

10 Extensive Games with Simultaneous Moves

10.1 Anyone Can Be a Leader (or Anti-Leader)

(Based on Myerson (2013), "Fundamentals of Social Choice Theory").

Two citizens, A and B, play a game where they simultaneously choose to Claim ($c_J = 1$) or not claim ($c_J = 0$). Any citizen who picks $c_J = 0$ gets utility 0, and when claiming gets utility $v > 0$ if the other does not claim and $-k < 0$ if the other does claim. Of course, there are two pure strategy NE to this game, one where $c_A = 1$ and $c_B = 0$, and one where $c_A = 0$ and $c_B = 1$.

i. Now suppose there is also a set of n "Leaders" $L_1, L_2, ..., L_n$. At the beginning of the game, the leaders simultaneously make a "law" by saying $s_i \in \{A, B\}$, which we will interpret as "player s_i gets to claim".

After the leaders make their choices, the citizens both observe $s = (s_1, ..., s_n)$, and then pick $c_J \in \{0, 1\}$ as above. Assume the citizen utilities are the same as before, i.e., they don't intrinsically care about what the leaders say. For now, assume the leader utility is $L_i = 0$, i.e., they get a utility of 0 no matter what.

i. Identify an SPNE to this game where the citizens do not condition their behavior on any of the leader choices, i.e., play the same strategy for any s.

ii. Show that for any $\hat{i}$, there is an SPNE to this game where $c_J^* = \mathbf{1}_{s_{\hat{i}}=J}$. How would you describe this SPNE in words?

iii. Show that for any $\hat{i}$, there is an SPNE to this game where $c_J^* = \mathbf{1}_{s_{\hat{i}} \neq J}$. How would you describe this SPNE in words?

iv. Now suppose the leaders prefer "being listened to". Formally:

$$u_{L_i} = \mathbf{1}_{c_{s_i}=1}$$

(We could also define being listened to as also requiring the other player not claim, though this will not affect the analysis.) Show that for any $\hat{\imath}$, there is still an SPNE to this game where $c^*_J = \mathbf{1}_{s_{\hat{\imath}}=J}$, or $c^*_J = \mathbf{1}_{s_{\hat{\imath}}\neq J}$. Be sure to specify what strategies the "other" leaders (i.e., not $\hat{\imath}$) use.

10.2 Avoiding Bargaining

Consider the following extensive game:

1. Players 1 and 2 simultaneously choose from $\{\text{In}, \text{Out}\}$
2. If either one of them chooses Out, the game ends with payoffs (y, y), where $0 < y < 1/2$
3. If both choose In:
   (a) Player 1 makes an offer $x \in [0, 1]$
   (b) Player 2 observes x and chooses $a \in \{0, 1\}$

If both players choose in, the utilities as a function of the offer and acceptance are:

$$u_1 = a(1 - x)$$
$$u_2 = ax$$

i. Find the SPNE to this game

ii. Identify another outcome which Pareto dominates the SPNE outcome(s)

iii. Now suppose $u_2 = ax + (1 - a)(1/2)$. What are the SPNE to this extensive game?

11 Repeated Games

11.1 Checking Bob's Claims

Here is a Normal Form Game from Powell's 1991 paper "Absolute and Relative Gains in International Relations Theory"[with what he calls T relabeled to B]:

		2: F	B	C
	F	$3, 3$	$1, 4$	$-1, 0$
1	B	$4, 1$	$2, 2$	$-1/2, 0$
	C	$0, -1$	$0, -1/2$	$0, 0$

The first two parts replicate one of the main points of the paper. (You can go read the paper if you are stuck!)

i. What are the NE to the (non-repeated) stage game?

ii. Now consider the twice-repeated version of the game with discount rate δ. Show that if δ is sufficiently high, there is an NE where both players choose F in the first round.

iii. When is there a SPNE to the twice-repeated version of this game where the first period outcome is (F, B)?

iv. For $0 < \delta < 1$, is there a SPNE to the twice-repeated version of this game where the first period outcome is (B, C) or (C, B)?

v. Show that for any $T > 2$, there is a SPNE to the T period version of this game where the outcome in periods $1, ..., T - 1$ is (F, F)

11.2 Dynamic guilt and cooperation

Here is a very simple two-period dynamic game. In both periods, two players choose among the strategy set $S_i^t = \{C, D\}$. In period 1, the payoffs are those of the standard

Prisoner's dilemma:

$$u_i^1 = \begin{cases} 3 & s_i^1 = D, s_{-i}^1 = C \\ 2 & s_i^1 = C, s_{-i}^1 = C \\ 1 & s_i^1 = D, s_{-i}^1 = D \\ 0 & s_i^1 = C, s_{-i}^1 = D \end{cases}$$

In period, two, the only difference is that if the other player cooperated, player i pays a cost $g > 1$ ("guilt") for defecting:

$$u_i^2 = \begin{cases} 3 - g\mathbf{1}_{s_{-i}^1 = C} & s_i^2 = D, s_{-i}^2 = C \\ 2 & s_i^2 = C, s_{-i}^2 = C \\ 1 - g\mathbf{1}_{s_{-i}^1 = C} & s_i^2 = D, s_{-i}^2 = D \\ 0 & s_i^2 = C, s_{-i}^2 = D \end{cases}$$

The payoff for the entire game is $U_i = u_i^1 + \delta u_i^2$ for $0 < \delta < 1$

i. In any SPNE, what are the equilibrium strategies in period 2?

ii. What is the SPNE outcome to this game if $\delta > 1/2$? If $\delta < 1/2$?

12 Infinitely Repeated Games

12.1 Asymmetric Infinite PD

Consider the infinite repetition of the following stage game:

		B	
		C	D
A	C	$2, x$	$0, 3$
	D	$3, 0$	$1, 1$

For $1 < x < 3$.

i. How would you interpret x?

ii. For what values of δ is both players using Grim Trigger 2 (i.e., cooperating unless either player has defected in the past) an SPNE for this game? How does x affect this? Interpret this result.

iii. Repeat part ii for the following game

		B	
		C	D
A	C	$2, 2$	$0, x$
	D	$3, 0$	$1, 1$

For $x > 2$.

12.2 Democracy and the rule of law

In this problem we will provide a more general (and complete) analysis of the game from Barry Weingast's 1997 paper "The political foundations of democracy and the rule of the law".

i. Here is the stage game. There is a ruler R and two groups, A and B. The ruler moves first and chooses who to steal from with choice set $s \in \{0, A, B, 2\}$, where 0 means stealing from no one, 2 means stealing from both groups, and A and B correspond to stealing from only that group. After the ruler decision, both groups observe what the ruler chose and then simultaneously choose to fight or not, where $f_J = 1$ means group J fights and $f_J = 0$ means they do not. Let $F = f_1 f_2$ be an indicator for whether **both** groups fight. The ruler payoff from the stage game is:

$$u_R = \begin{cases} 2v(1-F) - kF & s = 2 \\ v(1-F) - kF & s = A \text{ or } s = B \\ -kF & s = 0 \end{cases}$$

and for group J:

$$u_J = \begin{cases} v - cf_J & s \in \{0, -J\} \text{ or } F = 1 \\ -cf_J & \text{o/w} \end{cases}$$

In words, each group has property with value v. If the ruler does not steal from them *or* they are stolen from but both groups fight back, the group keeps v. Fighting costs c for each group, and if both fight the leader pays a cost k.

So, if both groups fight (regardless of the ruler choice), the ruler gets $-k$. If at least one group does not fight, the ruler gets $2v$ is stealing from both, v if stealing from one, and 0 if stealing from neither. The groups get v if retaining their property, and $-c$ if fighting.

Assume $c < v$ and $k < 0$. Find the pure strategy SPNE to the stage game (hint: there are 3).

ii. You should have found in part i that there is no SPNE where the leader steals from neither group ($s = 0$). The point of Weingast's paper is to show that in the repeated version of the game with common discount rate δ, this is possible.

In the repeated version, a strategy for the ruler is which group to steal from in each period t as a function of the history in all previous periods. The strategy for each group is whether to fight as a function of the history in all previous periods *and* what the ruler did the current period. Suppose the ruler uses the following trigger strategy:

$$s_R^t = \begin{cases} 0 & \text{no period } t' < t \text{ with } s^{t'} \neq 0 \text{ and } F = 0 \\ 2 & \text{o/w} \end{cases}$$

and the groups use the following strategy

$$f_J^t = \begin{cases} 1 & s_R^t \neq 0, \text{no period } t' < t \text{ with } s^{t'} \neq 0 \text{ and } F = 0 \\ 0 & o/w \end{cases}$$

ii. In words, what is the event which "triggers" the ruler to start stealing from both groups and the groups to accept this? Give an intuition for why this might be an effective way to prevent expropriation.

iii. Show that in any subgame with a previous period where $s^{t'} \neq 0$ and $F = 0$, the one deviation property holds (for each player) for any δ.

iv. Show that if there is no previous period with $s^{t'} \neq 0$ and $F = 0$, the one deviation property holds for the ruler for any δ (i.e., the ruler does not prefer to deviate to $s^{t'} \neq 0$ and then go back to playing the proposed strategy.)

iv. Finally, we need to show that the one deviation property holds for the groups if there is no previous period with $s^{t'} \neq 0$ and $\sum f_J^{t'} < 2$. Make an informal argument for why there is no incentive to deviate if the Ruler has chosen $s^t \in \{0, J, 2\}$ in the current period for any δ (formalize if you like), and more importantly show that if $\delta \geq \frac{c}{c+v}$ the one-deviation principle holds in subgames starting after $s^t = -J$ (i.e., the other group is stolen from).

12.3 Feasible Democratic Payoffs

Consider a game between parties L and R where in each stage nature picked one of them to win an election (L with probability p, R with probability $1 - p$), and the loser chooses to accept the result (giving the winner 1 and the loser 0) or fight (giving party L $q - c$ and party R $1 - q - c$).

i. To apply the folk theorem to this game, it is useful to first represent it as a normal form game. Each party's strategy is whether they accepted defeat $S_i = \{A, F\}$. Compute the expected utility for each of the four possible strategy profiles. For what values is (F, F) the unique NE?

ii. Set $p = q = 1/2$ and $c = 1/4$. Draw the appropriate quadrangle which shows the feasible average payoffs to the infinitely repeated game when the players are very patient ($\delta \to 1$), and the payoffs which can be sustained in an SPNE by the folk theorem (under the threat to play (F, F))

iii. How does the picture change if c increases? If p increases? If q increases? For each of these, draw a new figure and give a quick interpretation.

12.4 Feasible Treaties

In problem 9.4 a game where two countries (1 and 2) pick pollution levels $x_i \in [0, 1]$, with utility functions:

$$u_i = -(1 - x_i)^2 - x_{-i}^2$$

i. As shown in class, we can represent the outcomes of the game with a unit square, where player 1's "ideal" is $(1, 0)$ and player 2's ideal is $(0, 1)$, and the NE is at (1,1). Draw the set of outcomes which can be sustained by a "treaty grim trigger" strategy (i.e.,, emit at level x_i^* as long as both players chose (x_1^*, x_2^*) in all past periods) as $\delta \to 1$.

ii. What does the folk theorem say about the feasible SPNE payoffs as $\delta \to 1$ with the threat to revert to playing the NE? [Hint: You should get the same answer as part i, but these are not the same question!]

iii. Show that any average payoff (as $\delta \to 1$) which can be obtained in an SPNE to the infinitely repeated version of this game can be obtained in a SPNE where the emissions are the same in every period.

iv. As $\delta \to 1$, what is the highest payoff that country i can obtain in an SPNE to the infinitely repeated version of the game?

13 Simultaneous Games with Incomplete Information

13.1 PD with Uncertainty

Consider a Bayesian game that is a variation of the prisoner's dilemma. Both players are "altruistic" ($\theta_i = a$) with probability p and "bastards" ($\theta_i = b$) with probability $1 - p$. Both players learn their type and then simultaneously choose between C and D ($s_i \in \{C, D\}$).

The payoffs for the $\theta_i = b$ type are:

$$u_i(s, \theta_i = b) = \begin{cases} 3 & s_i = D, s_{-i} = C \\ 2 & s_i = C, s_{-i} = C \\ 1 & s_i = D, s_{-i} = D \\ 0 & s_i = C, s_{-i} = D \end{cases}$$

The payoff for type $\theta = a$ are:

$$u_i(s, \theta_i = a) = \begin{cases} 3 & s_i = D, s_{-i} = C \\ x & s_i = C, s_{-i} = C \\ 1 & s_i = D, s_{-i} = D \\ 0 & s_i = C, s_{-i} = D \end{cases}$$

For some $x > 3$.

a. Show that there is an BNE where both players always defect for both types.

b. For what values of x is there a BNE where the $\theta = a$ chooses C and the $\theta = b$ type chooses D? How does p affect the answer?

13.2 PD with Uncertainty and Correlated Types

In the previous problem, the probability that player 1 is altruistic is always p regardless of whether player 2 is altruistic. We now consider an example where player 2 is more likely to be altruistic when player 1 is altruistic.

Suppose the payoffs for the types are the same as above. To capture this suppose there is a state variable ω that is equal to 1 with probability 1/2 and equal to 0 with probability 1/2. The probability of being altruistic is now:

$$Pr(\theta_i = a|\omega) = \begin{cases} q & \omega = 1 \\ 1-q & \omega = 0 \end{cases}$$

for some $1/2 < q < 1$. So, the *unconditional* probability of being altruistic is

$$Pr(\theta_i = a) = (1/2)q + 1/2(1-q) = 1/2$$

a. Use Bayes' rule to find the probability that player 1 is altruistic given player 2 is altruistic. Is this greater than or equal to 1/2? Is it increasing or decreasing in q? Hint: we can write $Pr(\theta_2 = a, \theta_1 = a)$ as:

$$Pr(\theta_2 = a, \theta_1 = a) = Pr(\omega = 1)Pr(\theta_2 = a, \theta_1 = a|\omega = 1) + Pr(\omega = 0)Pr(\theta_2 = a, \theta_1 = a|\omega = 0)$$

b. For what values of x can cooperation be sustained in this version of the PD with uncertain types? Note that when $q = 1/2$ this problem is the same as problem 1 with $p = 1/2$. How does increasing q affect the minimum x required for cooperation?

14 Extensive Games with Incomplete Information

14.1 Perfect Bayesian Battle of the Berkeley Bowls

Recall we can represent a simultaneous move Battle of the Berkeley Bowls as an extensive game where player 1 moves first (choosing E or W), and then player 2 chooses E or W without observing 1's choice.

As before, if they choose different actions the payoffs are $(0, 0)$, if they both choose E the payoffs are $(2, 1)$, and if they both choose W the payoffs are $(1, 2)$.

Let p_i be the probability that player i plays E, and let q be the 2's belief about the probability that 1 has chosen E at the information set where she moves.

i. What is the sequentially rational strategy for player 2 (given her belief q)?

ii. What is the consistent belief for player 2 (given player 1's strategy)?

iii. What is the sequentially rational strategy for player 1 (given 2's strategy)

iv. Combining i-iii, what are the PBE assessments (i.e., strategies and beliefs) to this game?

14.2 Bargaining with Uncertainty over Cost

There are two players, 1 (the proposer) and 2 (the responder). First, nature determines whether player 2 is a "high cost" or "low cost" type. Player 2 observes her type but player 1 does not. Assume the prior probability of being a low cost type is p. Next, player 1 makes an offer x, $0 \leq x \leq 1$, and Player 2 accepts ($a = 1$) or rejects ($a = 0$). If player 2 accepts the payoffs are $(1 - x, x)$. If player 2 rejects, the players fight, giving expected payoffs $(q - c_1, 1 - q - c_\theta)$, where $q \in (0, 1)$ is the commonly known probability that player 1 wins. Assume $c_l < c_h$ (i.e., the low cost type pays a lower cost to fight) and that $c_\theta < 1 - q$ for $j = l, h$ (i.e., both types prefer fighting to getting nothing).

Find the PBE of this game. (You can assume player 2 accepts when indifferent.) Write an inequality that determines when conflict occurs, and interpret this expression.

14.3 Opposite Day and babbling

We typically look for PBE of cheap talk games where the informed player reports "honestly", but that is not the only kind of PBE to check for.

Consider a game with two players, an expert and decision-maker: $\{E, D\}$. Nature picks a binary state $\omega \in \{0, 1\}$, where $Pr(\omega = 1) = p$.

The expert observes a signal $s \in \{0, 1\}$ such that $Pr(\omega = 1|s) = p_s$, where $p_0 < p_1$. The expert then sends a message m, and the decision-maker, observing m, chooses policy x. The utilities are:

$$u_D = -(x - \omega)^2$$
$$u_E = -(x - (\omega + b))^2$$

where $b \geq 0$ is the *bias* of the expert.

The optimal choice for D given his belief is $x^*(m) = Pr(\omega = 1|m)$, and the expert prefers policy x_1 to policy x_2 (given her belief) if and only if:

$$EU_E(x_1; s) \geq EU_E(x_2; s) \Longrightarrow (x_1 - (p_s + b))^2 \geq (x_2 - (p_s + b))^2$$

i. Suppose there is a PBE where $m(s) = 1 - s$, i.e., the expert sends the opposite signal as her message. What are D's beliefs and sequentially rational policy choices as a function of m given this strategy?

ii. Show that it is optimal for the expert to follow this strategy given D's response if and only of $b \leq (p_1 - p_0)/2$.

iii. Now suppose there is a PBE where $m(s) = 1$ ("babbling"). What are D's beliefs (hint: one will be unconstrained!) and sequentially rational policy choices as a function of m given this strategy? Show that if the belief upon observing $m = 0$ is the same as upon observing $m = 1$, this messaging strategy can be part of a PBE.

14.4 Bargaining with continuous uncertainty over cost

Take the bargaining problem in 14.2, except now assume that the cost that 2 faces for fighting is drawn from a *uniform distribution* on $[\underline{c}, \overline{c}]$. I.e., each cost in that interval is equally likely. This can be represented by a probability distribution function:

$$Pr(c_2 \leq c) = \begin{cases} 0 & c < \underline{c} \\ \frac{c - \underline{c}}{\overline{c} - \underline{c}} & c \in [\underline{c}, \overline{c}] \\ 1 & c < \overline{c} \end{cases}$$

Again, 2 knows the cost c_2 while 1 does not, but does know this distribution.

i. What offers does 2 accept as a function of x and c_2? What is the probability that an offer x is accepted (hint: it will be 0 if x is sufficiently low, 1 if x is sufficiently high, and between 0 and 1 for intermediate x)?

ii. What is the expected utility for player 1 to make offer x, given 2's sequentially rational strategy?

iii. Your answer to part ii should indicate that the expected utility is quadratic in x for the middle interval. What is the maximizing offer of the function on this interval?

iv. If the maximizing offer from part iii lies on the middle interval, then this will be the offer which maximizes the overall utility function (why?) Come up with a condition for when this is true. Does conflict happen when it is not true? When it is true?

14.5 Bargaining with Binary Types

Two countries are bargaining over territory. Country 1 will make an offer for how to divide the territory $x \in [0, 1]$, which country 2 can accept or reject. If rejecting, they fight a war with expected payoffs $(q_\theta - c, 1 - q_\theta - c)$.

Country 2 has been developing a new military technology. Let θ represent the success of their project: if $\theta = h$ ("high type") it has succeeded, and if $\theta = l$ ("low type") it has not.

With perfect information, the order of moves is:

1. Nature picks $\theta = h$ with probability p and $\theta = l$ with probability $1 - p$
2. Country 1 (knowing θ) makes an offer $x \in [0, 1]$
3. Country 2 observes x (and θ) and chooses to accept or reject

The payoffs are:

$$u_1 = a(1 - x) + (1 - a)(q_\theta - c)$$
$$u_2 = ax + (1 - a)(1 - q_\theta - c)$$

Assume $c > 0$, $q_h < q_l$ (Why?), $q_\theta - c > 0$, and $1 - q_\theta - c > 0$

i. With perfect information, what is the PBE of this game (hint: since there is complete information, this is the same as asking for SPNE, and we need not specify any beliefs.)

ii. Now suppose Country 1 does not know θ (but Country 2 does). Now what is the PBE to the game? (Hint: find a threshold c such that determines whether country 1 makes an offer that both type accept versus one that only one type accepts.)

iii. Suppose before making the offer, Country 1 can pay a cost $k > 0$ to hire a spy to learn θ. I.e., if not paying, the game proceeds as in part ii, and if paying the game proceeds as in part i (with a $-k$ added to Country 1's utility). For what values of k will country 1 hire the spy?

iv. Now consider a model with the same payoffs and bargaining protocol as in parts i-ii (i.e., with no spy option), but country 1 knows θ while country 2 does not. A natural equilibrium to check for is one where country 1 plays the same strategy as in part i, and 2 accepts both offers made in equilibrium. Show that given this strategy is sequentially rational for country 2 when they form consist beliefs, but, player 1 has an incentive to deviate, so this is not a PBE.

14.6 Educational Signaling

Here is a version of the Spence educational signaling model, where the actors are a Worker (W) and a Firm (F):

- Nature chooses the worker productivity $\theta \in \{l, h\}$, with $Pr(\theta = h) = p$
- The worker observes θ and chooses an education level $e \geq 0$
- The firm observes e (but not θ) and chooses a wage $w \geq 0$

The utility functions are:

$$u_W = w - c_\theta e$$

$$u_F = -(w - x_\theta)^2$$

where c_θ is the marginal cost of education for type θ, and x_θ is the productivity of a worker with type θ. Assume $c_l > c_h$ and $x_l < x_h$. To streamline algebra, normalize the productivity of the low type to 0: $x_l = 0$.

Let e_l be the education level chosen by the low type, e_h the level chosen by the high type, and $\hat{p}(e)$ be the firm's belief about the probability the worker is a high type as a function of

e.

i. Show that given $\hat{p}(e)$, sequential rationality for the firm implies the wage offered is $\hat{p}(e)x_h$.

ii. First let's check for a *separating* equilibrium where $e_l \neq e_h$. In such an equilibrium, what does consistency imply about the belief of the firm and hence the wage offered for all e (hint: for "most" e this will be indeterminate)?

iii. Given ii, show that there is no separating equilibrium where $e_l > 0$.

iv. Show that there exists a $\underline{e}$ and $\overline{e}$, $0 < \underline{e} < \overline{e}$ such that there is a separating PBE where $e_l = 0$ and $e_h \in [\underline{e}, \overline{e}]$. (hint: set the beliefs for any e other than e_l and e_h such that no worker wants to pick these levels, and then check that the low type wants to send $e = 0$ rather than e_h *and* the high type would rather send e_h than $e = 0$)

v. Now let's check for a pooling equilibrium where $e_l = e_h$, write the "common" message e_c. In this proposed equilibrium, what is the firm belief and wage set upon observing e_c?

vi. What is the highest level of e_c such that there is a PBE where both types choose this education level? (Be sure to specify the off-path beliefs!)

vii. We have identified multiple separating and pooling equilibria. What are the equilibrium payoffs to the worker (of each type) and firm in these equilibria? Which is the "best" equilibrium for each worker type and the firm?